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ON FARTHEST POINTS IN METRIC SPACES

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ABSTRACT. For a non-empty bounded subset K of a metric space (X, d) and $x \in X$, a point $k_o \in K$ is called a farthest point to x in K if $d(x, k_o) = \sup\{d(x, k) : k \in K\}$. The K is said to be remotal if each $x \in X$ has a farthest point in K, and it is said to be uniquely remotal if each $x \in X$ has a unique farthest point in K. The map which takes each element $x \in X$ to its set of farthest points in K is called the farthest point map. In this paper, we discuss some results on farthest points, remotal sets, uniquely remotals sets and on the farthest point map. The underlying spaces are metric spaces and convex metric spaces. The results of this paper generalize and extend some known results on the subject. Some directions for further research on the topic have also been given in the paper.

1. INTRODUCTION

A basic problem in the theory of farthest points is: Given a nonempty bounded subset K of a metric space (X, d) and $x \in X$, find a $k_o \in K$ such that $d(x, k_o) = \sup\{d(x, k) : k \in K\} \equiv \delta(x, K)$. Such a point $k_o \in K$ may or may not exist. Even if it exists, it may or may not be unique. The set K is said to be remotal if such a k_o exists for each $x \in X$ and is called uniquely remotal if such a k_o uniquely exists for each $x \in X$. The map which takes each $x \in X$ to the set of its farthest points is called the farthest point map. Farthest points are important building blocks of convex sets which are extensively applied in programming. These points are widely used in the study of extremal structure of sets, methods of finding exposed points of sets. The study of remotal and uniquely remotal sets has attracted many researchers due to their connection with geometry of Banach spaces.

One of the natural trends of mathematical research is to refine the framework of the known results and to examine which of the results available in

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normed linear spaces survive in more general spaces. For most of the literature in the theory of farthest points, the underlying spaces are either Hilbert spaces or normed linear spaces. The construction of farthest point theory in more abstract spaces is a challenging one. Some attempts have already been made in this direction and the present paper is also a step in the same direction. In this paper, we discuss some results on farthest points, on remotal sets, uniquely remotal sets and on the farthest point map. The spaces considered are metric spaces or convex metric spaces. The proved results generalize and extend some known results on the subject. Some directions for further research on the topic have also been given in the paper.

2. NOTATIONS AND DEFINITIONS

In this section, we set up some notations and recall few definitions needed in the subsequent sections.

Throughout, $\mathbb{R}$ stands for set of real numbers, $\mathbb{N}$ for natural numbers, iff for if and only if, s.t.b. for said to be, i.e. for that is, f.p.p for farthest point problem, f.p.m. for farthest point map. $S[x, r]$ stands for a sphere with center x and radius r.

We now recall, few definitions :

A mapping $W : X \times X \times [0, 1] \to X$ is s.t.b. a **convex structure** on a metric space (X, d) if for all $x, y, u \in X$ and $\lambda \in [0, 1]$, $d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$, The metric space (X, d) together with a convex structure W is called a **W- convex metric space** or **Takahashi convex metric space**[18], and is denoted by (X, d, W).

Takahashi idea of convex structure on a metric space is a natural generalization of convexity in normed linear spaces. Every normed linear space $(X, \|.\|)$ is a W- convex metric space with convex structure W defined by $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$. But there are plenty of Takahashi convex metric spaces which are not normed linear spaces. The set $\mathbb{R}^2$ with the river metric d^* defined by

$$d^*(v_1, v_2) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2 \\ |y_1| + |y_2| + |x_1 - x_2|, & x_1 \neq x_2, \end{cases}$$

$v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in \mathbb{R}^2$ and with convex structure $W : \mathbb{R}^2 \times \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2$ defined by $W(v_1, v_2, \lambda) = z = (z_x, z_y)$ where $d^*(z, v_1) =$

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$(1-\lambda)d^*(v_1, v_2)$, $d^*(z, v_2) = \lambda d^*(v_1, v_2)$ is a complete Takahashi convex metric space, which is not a normed linear space and not even a metric linear space (see[13]).

If (X, d) is a complete metric space, we say that the space X admits **mid points** (see [6]) if for every $x, y \in X$ there is an $m(x, y)$ such that $d(x, m(x, y)) = d(y, m(x, y)) = \frac{1}{2}d(x, y)$.

It is easy to see (see [18]) that $d(x, W(x, y, \frac{1}{2})) = d(y, W(x, y, \frac{1}{2}) = \frac{1}{2}d(x, y)$.

A real-valued function f on a convex metric space (X, d, W) is s.t.b. **W-convex** [12] if for all $x, y \in X$, $t \in [0, 1]$ $f(W(x, y, t)) \leq (1-t)f(x) + tf(y)$.

A convex metric space (X, d, W) is s.t.b. **strictly convex**(see [12]) if for all $x, y, z \in X$, $x \neq y$, and $r > 0$, $d(x, z) \leq r$, $d(y, z) \leq r$ imply $d(W(x, y, \frac{1}{2}), z) < r$.

The river metric space $(\mathbb{R}^2, d^*, W)$ is strictly convex[13].

A linear space X equipped with a metric d is called a **metric linear space** if both addition and scalar multiplication in X are continuous and the metric d is translation invariant i.e. $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$.

It is easy to see that every normed linear space is a metric linear space but there are plenty of metric linear spaces which are not normed linear spaces (see [16]).

A metric linear space (X, d) is s.t.b. **strictly convex** [2] if for any $r > 0$, $x, y \in X$, $x \neq y$,, $d(x, 0) \leq r$, $d(y, 0) \leq r$ imply $d(\frac{x+y}{2}, 0) < r$.

The strict convexity of Takahashi convex spaces and of metric linear spaces are natural generalizations of well known notion of strict convexity of normed linear spaces.

Example[16] Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} \frac{2t}{1+t}, & \text{if } 0 \leq t \leq 1 \\ t, & \text{t} > 1. \end{cases}$$

and d be the linear metric on $\mathbb{R}$ defined by $d(s, t) = f(|s-t|)$ for all $s, t \in \mathbb{R}$, then $(\mathbb{R}, d)$ is a strictly convex metric linear space.

For a non-empty bounded subset K of a metric space (X, d) and $x \in X$, a sequence $< k_n >$ in K is called a maximizing sequence for x in K

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if $\lim d(x, k_n) = \delta(x, K)$. The set K is s.t.b. **nearly compact** or **M-compact** (see [15]) if for any $x \in X$, every maximizing sequence $< k_n >$ in K has a convergent subsequence in K.

Clearly, every compact set is nearly compact. However, (see [15]), a nearly compact set need not be compact , it may not even be closed (see [15]). It is well known (see [15]) that a nearly compact set is always remotal.

3. Main Results

In this section, we discuss main results of the paper and also give some directions for further research on the topic.

Let W be a non-empty bounded subset of a metric space (X, d), $x \in X$ and $F_W(x) = \{g \in W : d(x, y) = \delta(x, W)\}$. Suppose $g \in W$ and $F_g = \{x \in X : g \in F_W(x)\}$. Concerning the set F_g, we have

Theorem 3.1. *Suppose W be a non-empty bounded subset of a metric space (X, d), then*

(1) For every $g \in W$, F_g is a closed set if W is a closed set,
(2) $X = \underset{g \in W}{\cup} F_g$ if W is remotal.

Proof. (1) Since $F_g = \{x \in X : g \in F_W(x)\} = W \bigcap S[g, \delta(x, W)]$, the result follows.

(2) Since $F_g = \{x \in X : g \in F_W(x)\} \subseteq X$ for every $g \in W$, , $\underset{g \in W}{\cup} F_g \subseteq X$.

Now, suppose $x \in X$. Since W is remotal , there exists some $g \in W$ such that $g \in F_W(x)$ i. e. $x \in F_g$ for some $g \in W$, and so $X \subseteq \underset{g \in W}{\cup} F_g$. Hence $X = \underset{g \in W}{\cup} F_g$.

□

Remark 3.1. For normed linear spaces, these results were proved in [10].

Some basic properties of the farthest point mapping are given in the following theorem:

Theorem 3.2. *let K be a non-empty bounded subset of a metric space (X, d), then*

(1) $|d(x, F_K(x)) - d(y, F_K(y))| \leq d(x, y)$ for all $x, y \in X$,
(2) $d(x, y) \leq d(x, F_K(x)) + d(y, F_K(y))$,

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(3) *If K_1 is a non-empty subset of K, then $d(x, F_K(x)) \geq d(x, F_{K_1}(x))$ for all $x \in D(F_K) \cap D(F_{K_1})$, where $D(F_K)$ denotes the domain of the farthest point map $F_K : D_{F_K} \to K$ i.e. $D(F_K) = \{x \in X : F_K(x) \neq \phi\}$.*
(4) *For $k_o \in K$, the set $F_K^{-1}(k_o) = \{x \in X : k_o \in F_K(x)\}$ is a closed set.*

For normed linear spaces, Theorem 3.2 was proved by Elumalai and Vijayaragava2 n [4] and the proof given in [4] for normed linear spaces can be easily extended to metric spaces.

Example 3.1. [4] Let $X = \mathbb{R}^2$ with the Euclidean norm and $K = \{(x_1, x_2) : 0 \leq x_1 \leq 2, -1 \leq x_2 \leq 1\}$. If $x = (2, 2)$, then $(0, -1) \in F_K(x)$. If $x = (1, 2)$, then $(0, -1)$ and $(2, -1)$ are both in $F_K(x)$. Here $D(F_K) = \mathbb{R}^2$.

Example 3.2. [3] Let $X = l_1 = \{x =< x_n >: \sum_{n=1}^{\infty} | \ x_n| < \infty\} \equiv$ the space of all summable sequences, and $K = \{x =< x_n >\in l_1 : \sum_{n=1}^{\infty} | \ x_n| + |x_n|^2 \leq 1\}$, then $D_K(x) = \phi$. Such sets K are called anti-remotal sets.

In the theory of nearest points, the following result is well known (see [8]):

If A is a compact subset of a metric space (X, d), then for any subset B of X, there is a $p \in A$ such that $d(p, B) = d(A, B)$.

A natural question that arises is : whether an analogous result holds in the theory of farthest points? The following theorem shows that this is true.

Theorem 3.3. *If A is a compact subset of a metric space (X, d), then for any bounded subset B of X, there is a $p \in A$ such that $\delta(p, B) = \delta(A, B)$.*

Proof. Let $\delta(A, B) = r$. Since $\delta(A, B) = \sup\{d(x, y) : x \in A \text{ and } y \in B\}$, for every positive integer n, there exists $a_n \in A$ and $b_n \in B$ such that

$$r \geq d(a_n, b_n) > r - \frac{1}{n}.$$

Since A is compact, the sequence $< a_n >$ has a subsequence $< a_{n_i} >\to p \in A$. We claim that $\delta(p, B) = \delta(A, B) = r$. Suppose $\delta(p, B) < r$, say $\delta(p, B) = r - \varepsilon$, $\varepsilon > 0$. Since $< a_{n_i} >\to p$, for $\frac{\varepsilon}{2} > 0$, there exists $m_o \in \mathbb{N}$ such that $d(p, a_{n_i}) < \frac{\varepsilon}{2}$ for every $n \geq m_o$. Choose $n_o \geq m_o$ such that $\frac{1}{n_o} < \frac{\varepsilon}{2}$

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i.e. $n_o > \frac{2}{\varepsilon}$. So $d(p, a_{n_o}) < \frac{\varepsilon}{2}$ and $d(a_{n_o}, b_{n_o}) > r - \frac{1}{n_o} > r - \frac{\varepsilon}{2}$. Then

$$\begin{aligned} d(p, b_{n_o}) &\geq d(a_{n_o}, b_{n_o}) - d(p, a_{n_o}), \\ &> r - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \\ &= r - \varepsilon \\ &= \delta(p, B) \end{aligned}$$

i. e. $d(p, b_{n_o}) > \delta(p, B)$, a contradiction. Hence $\delta(p, B) = r = \delta(A, B)$ $\square$

Taking $B = \{x\}$, $x \in X$, we obtain that for each $x \in X$ there exists $p \in A$ such that $d(p, x) = \delta(A, x)$ i.e. each $x \in X$ has farthest point in A.

Corollary 3.1. Every compact subset of a metric space is remotal.

The following example shows that any subset of a convex metric space (X, d, W) containing two elements can not be uniquely remotal.

Let $K = \{x, y\}$, $x \neq y \in X$. Suppose K is uniquely remotal. Let $z = W(x, y, \frac{1}{2}) \in X$. Then $d(z, x) = d(W(x, y, \frac{1}{2}), x) = \frac{1}{2}d(x, y)$, $d(z, y) = d(W(x, y, \frac{1}{2}), y) = \frac{1}{2}d(x, y)$. Also,

$$\delta(z, K) = \sup\{d(z, x), d(z, y)\} = \frac{1}{2}d(x, y)$$

So, $d(z, x) = \delta(z, K)$, $d(z, y) = \delta(z, K)$ i.e. both x and y are farthest points to $z \in X$ in K. Hence K is not uniquely remotal.

Proceeding as above, one can prove the following:

In a metric space, (X, d) admitting mid-points, no subset of X containing two points can be uniquely remotal.

For a bounded subset K of a metric space (X, d), the function $r_K : X \to [0, \infty[$ defined by $r_K(x) = \delta(x, K) \equiv \sup\{d(x, y) : y \in K)\}$ is called the radial function for K. The number $r_K(x)$ is the radius of the smallest ball in X with center x and containing K.

It is known (see [14]) that for a remotal set K in a metric space (X, d), the function r_K is non-expansive and hence uniformly continuous. If the space X is a normed linear space, then the function r_K is convex (see [5], [7]). We extend the result to convex metric spaces(see also [14]).

Theorem 3.4. *If K is a non-empty bounded subset of a convex metric (X, d, W), then the radial function r_K is convex.*

To prove the theorem, we need the following lemma:

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Lemma 3.1. [1] *The pointwise supremum of W-convex functions is W-convex.*

Proof. Let $\{f_i : i \in I\}$ be a family of W-convex functions from a convex space (X, d, W) to $\mathbb{R}$ and $f = \sup\limits_{i \in I} f_i$ i.e. $f_i(x) \leq f(x)$ for all $x \in X$ Then for $i \in I$

$$\begin{aligned} f_i[W(x, y, \lambda)] &\leq (1 - \lambda)f_i(x) + \lambda f_i(y) \\ &\leq (1 - \lambda)f(x) + \lambda f(y) \end{aligned}$$

This implies $\sup\limits_{i \in I} f_i[(W(x, y, \lambda))] \leq (1 - \lambda)f(x) + \lambda f(y)$ i.e.

$$f[W(x, y, \lambda)] \leq (1 - \lambda)f(x) + \lambda f(y)$$

i.e. f is W-convex. □

Proof of the Theorem: For each $k \in K$, let $g_k : X \to [0, \infty[$ be defined by $g_k(x) = d(x, k)$. Then each g_k is W-convex as

$$\begin{aligned} g_k[W(x_1, x_2, t)] &= d(W(x_1, x_2, t), k) \\ &\leq (1 - t)d(x_1, k) + td(x_2, k) \\ &= (1 - t)g_k(x_1) + tg_k(x_2) \end{aligned}$$

Now for each $x \in X$, $r_K = \sup\limits_{k \in K} g_k(x)$, and since by Lemma 3.1 the pointwise supremum of an arbitrary collection of W-convex functions is W-convex, we obtain that the radial function r_K is W-convex.

Let A and B be non-empty bounded subsets of a metric space (X, d) and $\delta(A, B) = \sup\{d(x, y) : x \in A, y \in B\}$. The pair (A, B) is said to have FP**-property** [9] if $d(x_1, y_1) = \delta(A, B)$ and $d(x_2, y_2) = \delta(A, B)$ imply $d(x_1, x_2) = d(y_1, y_2)$ for every $x_1, x_2 \in A$ and $y_1, y_2 \in B$. The metric space X has FP-property if every pair (A, B) of non-empty bounded subsets of X has FP-Property.

Let $A^o = \{x \in A : d(x, y) = \delta(A, B)$ for some $y \in B\}$ and $B^o = \{y \in B : d(x, y) = \delta(A, B)$ for some $x \in A\}$.

The following theorems give conditions under which $A^o \neq \phi$, $B^o \neq \phi$.

Theorem 3.5. *If A is compact subset of a metric space (X, d) and B is a bounded subset of X which is nearly compact w.r.t. A, then A^o and B^o are non-void.*

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Proof. Let $x \in A$. Since the set B is nearly compact w.r.t. A, it is remotal w.r.t A and so, the function $\delta(x, B)$ is continuous on A. Since A is compact, it attains its supremum $\delta(A, B)$ at some point $x_o \in A$, i.e.

$$\delta(x_o, B) = \sup_{x \in A} \delta(x, B) = \delta(A, B).$$

By the definition of $\delta(x_o, B)$, there is a sequence $< y_n >$ in B such that $\delta(x_o, y_n) \to \delta(x_o, B)$ i.e. $< y_n >$ is a maximizing sequence for x_o in B. Since the set B is nearly compact w.r.t. A, $< y_n >$ has a subsequence $< y_{n_i} > \to y_o \in B$. Then

$$d(x_o, y_o) = \lim \delta(x_o, < y_{n_i} >) = \delta(x_o, B) = \delta(A, B).$$

Since $x_o \in A^o$ and $y_o \in B^o$, we obtain the desired result. □

Remark 3.2. The proof of the theorem shows that the above result is true if B is remotal w.r.t. the set A and the radial function $\delta(x, B)$ attains its supremum at some point of A.

Using FP-property, we prove the following:

Theorem 3.6. *If K is a non-empty remotal subset of a metric space (X, d) having FP-property, then K is uniquely remotal.*

Proof. Let $x \in X$ be arbitrary. Since K is remotal, there exists $k_o \in K$ such that $d(x, k_o) = \delta(x, K)$. Suppose there exists $k_1 \in K$ such that $d(x, k_1) = \delta(x, K)$. Let $A = \{x\}$, $B = K$, then $d(x, k_o) = \delta(\{x\}, K)$. Since $d(x, k_1) = \delta(\{x\}, K)$ and X has FP-property, we have $d(k_o, k_1) = d(x, x) = 0$. This gives $k_o = k_1$, and hence K is uniquely remotal. □

Remark 3.3. Analogous to the (P)-property introduced and discussed by Raj and Eldred[17] , FP-property was introduced in [9] to discuss some results on farthest points. The question of the existence of pair of sets having FP-property, and also of metric spaces having FP-property is doubtful and remains to be investigated.

The existence of farthest points of K is equivalent to the fact that the set $D = \{x \in X : \text{ there exist } k_o \in K, d(x, k_o) = r_K(x)\}$ is non-empty.

Diville and Zizler [3] proved the following:

Let X be a strictly convex Banach space and K a closed bounded subset of X such that the corresponding set D is dense in X . Then the set $U = \{y \in X : y \text{ has a unique farthest point in } K\}$ is also dense in X.

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It will be interesting to extend this result to strictly convex metric linear spaces and strictly convex Takahashi spaces.

Let us consider the quantitative version of the set of farthest points:

Let $\varepsilon > 0$ and K be a non-empty bounded subset of a metric space (X, d). We say that K is an **ε-uniquely remotal** set if K is remotal and $diam\ F_K(x) \leq \varepsilon$ for each $x \in X$.

A study of such sets was taken up by Moors [11] who took a quantitative look of the f.f.p.

It was shown in [11] that under suitable conditions ε-uniquely remotal sets are almost singleton sets (i.e. have diameter of atmost 2ε).

It may be remarked that uniquely remotal sets are directly related to Chebyshev sets (see [5], [11]): there exists a non-convex Chebyshev set of a Hilbert space iff there exists a non-trivial set (i.e. a non-singleton set) that is uniquely remotal.

Among other results, the following results were proved in [11]:

(i) If K is an ε-uniquely remotal subset of a Banach space X and the farthest point map is upper semi-continuous, then $diam\ K \leq 2\varepsilon$.

(ii) If K is compact ε-uniquely remotal subset of a Banach space X, then $diam\ K \leq 2\varepsilon$. In particular, every ε-uniquely remotal subset K of a finite dimentional Banach space has $diam\ K \leq 2\varepsilon$.

A natural question that arises is : Can we replace the conclusion that $diam\ K \leq 2\varepsilon$ by $diam\ K \leq s\varepsilon$, where $1 \leq s < 2$? Does it depend upon the norm ? It will also be interesting to extend these results to spaces more general than Banach spaces.

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